Implicit Function Theorem

Theorem 168. Let $U \subseteq \mathbb{R}^{n+1}$ be an open neighbourhood. Suppose $F(\mathbf{x}, y)$ is a C^1 function on U and $(\mathbf{a}, b) \in U$ satisfies $F(\mathbf{a}, b) = 0$. If $\partial_y F(\mathbf{a}, b) \neq 0$, then there exists an open set $V \subseteq \mathbb{R}^n$ together with a unique C^1 function $f : V \to \mathbb{R}$ such that $F(\mathbf{x}, f(\mathbf{x})) = 0$ for all $x \in V$. Moreover, the derivative of f is given by

$$\partial_{x_i} f(\mathbf{x}) = -\frac{\partial_{x_i} F(\mathbf{x}, y)}{\partial_y F(\mathbf{x}, y)}.$$

1. Argue that you can find $r, \rho > 0$ such that $F(\mathbf{x}, b-\rho) < 0$ for all $x \in B_r(\mathbf{a})$ and $F(\mathbf{x}, b+\rho) > 0$ for all $x \in B_r(\mathbf{a})$.
2. Use (1) to prove that for any fixed $\mathbf{x}_0 \in B_r(\mathbf{a})$, there exists a unique point $y_0 \in B_\rho(b)$ such that $F(\mathbf{x}_0, y_0) = 0$. Define $f(\mathbf{x}_0) = y_0$.
3. For each $i \in \{1, \dots, n\}$, let $k_i = f(\mathbf{x}_0 + t\mathbf{e}_i) - f(\mathbf{x}_0)$ for some sufficiently small t, so that $y_0 + k_i = f(\mathbf{x}_0 + t\mathbf{e}_i)$. Argue that $F(\mathbf{x}_0 + t\mathbf{e}_i, y_0 + k) = F(\mathbf{x}_0, y_0) = 0$ and apply the Mean Value Theorem. Re-arrange your result to show that $\partial_i f$ exists and is continuous.

Theorem 169. Let U be an open subset of $\mathbb{R}^{n+k}$ and $\mathbf{F} : U \to \mathbb{R}^k$ be a C^1 function. Write $(x_1, \dots, x_n, y_1, \dots, y_k)$ for the coordinates in $\mathbb{R}^{n+k}$. If $(\mathbf{a}, \mathbf{b})$ satisfies $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ and $B_{ij} = \left(\partial_{y_j} F_i(\mathbf{a}, \mathbf{b})\right)_{ij}$ is invertible, there exists an open set $V \subseteq \mathbb{R}^n$ and a unique C^1 function $\mathbf{f} : V \to \mathbb{R}^k$ such that $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = 0$ for all $\mathbf{x} \in V$.

Proceed by induction on the number of variables $(y_1, \dots, y_k)$. The base case is Theorem 168, so assume the result holds in $k - 1$ variables.

1. Suppose B is an invertible matrix, and let B_{ij} denote B with the ith row and jth column deleted. Argue that there must be a $(k-1) \times (k-1)$ submatrix that is invertible. Argue that you can assume B_{kk} is invertible.
2. Apply the induction hypothesis to write $(y_1, \dots, y_{k-1}) = \mathbf{f}(\mathbf{x}, y_k)$ for some function $\mathbf{f}$.
3. Define $G(\mathbf{x}, y_k) = F_k(\mathbf{x}, \mathbf{f}(\mathbf{x}, y_k), y_k)$. Argue that it's sufficient to show that $\partial_{y_k} G(\mathbf{a}, b_k) \neq 0$.
4. The hard part is actually showing that $\partial_{y_k} G(\mathbf{a}, b_k) \neq 0$.
   (a) Apply the chain rule to find a formula for $\partial_{y_k} G(\mathbf{x}, y_k)$.
   (b) Differentiate the equation $F_i(\mathbf{x}, \mathbf{f}(\mathbf{x}, y_k), y_k) = 0$ with respect to y_k. Solve this equation for $\partial_{y_k} f_i$ by using Cramer's Rule.
   (c) Conclude that $\partial_{y_k} G(\mathbf{a}, b_k) \neq 0$.

Problem 170. Consider the non-linear system of equations:

$$\begin{aligned} xu^2 + yv^2 &= 9 \\ xv^2 - yu^2 &= 7 \end{aligned}$$

Find conditions which guarantee that we can write u and v as functions of x and y, and compute the partial derivatives of u and v with respect to these variables.

Theorem 171. Let $U, V \subseteq \mathbb{R}^n$ and fix some point $\mathbf{a} \in U$. If $\mathbf{f} : U \to V$ is of class C^1 and $D\mathbf{f}(\mathbf{a})$ is invertible, then there exists neighbourhoods $\tilde{U} \subseteq U$ of $\mathbf{a}$ and $\tilde{V} \subseteq V$ of $\mathbf{f}(\mathbf{a})$ such that $\mathbf{f}|_{\tilde{U}} : \tilde{U} \to \tilde{V}$ is bijective with C^1 inverse $(\mathbf{f}|_{\tilde{U}})^{-1} : \tilde{V} \to \tilde{U}$. Moreover, if $\mathbf{b} = \mathbf{f}(\mathbf{a})$ then the derivative of the inverse map is given by

$$[D\mathbf{f}^{-1}](\mathbf{b}) = [D\mathbf{f}(\mathbf{a})]^{-1}.$$

Hint: Define a function $\mathbf{F} : U \times V \to \mathbb{R}^n$ by $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \mathbf{f}(\mathbf{x})$.

Problem 172. If $U, V \subseteq \mathbb{R}^n$, a map $\mathbf{f} : U \to V$ is said to be a *diffeomorphism* if $\mathbf{f}$ is bijective and both $\mathbf{f}$ and $\mathbf{f}^{-1}$ are C^1.

1. Show that if $\mathbf{f} : U \to V$ is a C^1 homeomorphism such that $\det D\mathbf{f}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in U$, then $\mathbf{f}$ is in fact a diffeomorphism.
2. Show that the requirement that $\det D\mathbf{f}(\mathbf{x}) \neq 0$ is necessary by giving an example of a C^1 homeomorphism which is not a diffeomorphism.

Problem 173. Define the set

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

to be the set of 2×2 matrices. Define a map $g : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $g(A) = A^2$. Determine whether g is invertible in a neighbourhood of $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.